RATIONALITY OF MOTIVIC CHOW SERIES MODULO $\mathbb{A}^1$-HOMOTOPY

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ABSTRACT. Consider the formal power series $\sum[C_{p,\alpha}(X)]t^\alpha$ (called Motivic Chow Series), where $C_p(X) = \amalg C_{p,\alpha}(X)$ is the Chow variety of X parametrizing the p-dimensional effective cycles on X with $C_{p,\alpha}(X)$ its connected components, and $[C_{p,\alpha}(X)]$ its class in $K(\mathcal{C}hM)_{\mathbb{A}^1}$, the K-ring of Chow motives modulo $\mathbb{A}^1$ homotopy. Using Picard product formula and Torus action, we will show that the Motivic Chow Series is rational in many cases.

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0. INTRODUCTION

Let X be an algebraic variety over a finite field $\mathbb{F}_q$. André Weil [17] conjectured that the formal power series $\sum_d |Sym^d(X)(\mathbb{F}_q)|t^d$ is a rational function, in this paper we call this series as the Weil zeta series. This is part of what it is known as the Weil conjecture and was proved by Dwork in 1960, see [4]. Weil also observed that if a suitable cohomology theory exists, axiomatized as Weil cohomology, then the rationality of the Weil zeta series follows. Furthermore, the degree of the denominator is the dimension of the odd part of the cohomology ring, and the degree of the numerator is the dimension of the even part.

In 2000, Kapranov [10] proved that if X is a smooth projective curve, then the series $\sum[Sym^d X]t^d$, that we call Kapranov motivic series, is a rational

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function in $K'(\mathcal{V}ar/\kappa)[[t]]$, where $K'(\mathcal{V}ar/\kappa)$ is the K-ring of algebraic varieties over a fixed field κ, together with the relation $[X] = [C] + [X - C]$ where $C \subset X$ is a closed subscheme. The rationality of the Kapranov motivic zeta series implies the rationality of the Weil motivic zeta series when the base field is a finite field. Moreover, it makes sense to ask if the Kapranov motivic zeta is rational for a general variety, even when the base field κ is an infinite field.

In 2004, Larsen and Lunts [12] proved that when X is a surface, the Kapranov motivic zeta series $\sum [Sym^d X] t^d$ is rational in $K'(\mathcal{V}ar/\kappa)[[t]]$ if and only if X is a ruled surface. This tells us that we can only expect the rationality of the Kapranov motivic zeta in $K'(\mathcal{V}ar/\kappa)$ for special varieties.

Then in 2005 Y. André [1] observed that if the motive of X is finite dimensional, then the formal power series $\sum [Sym^d X] t^d$ in the ring of formal power series with coefficients in the K-ring of Chow motives, denoted by $K(\mathcal{C}hM)[[t]]$, is rational, as before we call this series the André motivic zeta series. The Chow motives of products of curves and Abelian varieties are finite dimensional [11], in particular some non-ruled surfaces. Moreover, if one assumes Bloch-Beilinson conjecture, then all Chow motives are finite dimensional, so André motivic zeta is conjecturally always rational.

Let $X \subset \mathbb{P}^n$ be a projective variety, and $C_{p,d}(X)$ be the Chow variety of X, which parametrizes the effective p-cycles on X with degree d. In particular, the Chow variety $C_{0,d}(X)$ of zero cycles of degree p parametrizes a formal linear combination of points $P_1 + P_2 + \cdots + P_d$ on X, hence the d-th symmetric product $Sym^d X$ is canonically identified with $C_{0,d}(X)$. In [5], the first author proves that the formal power series, the Euler-Chow series, $\sum [\chi(C_{p,d}(X)] t^d$ is rational for any simplicial projective toric variety, where $\chi(Y)$ is the Euler characteristic of Y.

Observing these phenomena, it would be natural to ask if $\sum [C_{p,d}(X)] t^d$ is rational in $K(\mathcal{C}hM)[[t]]$. In a previous paper by the authors [6], it was proved that if $n \geq 2$ then the series

$$\sum [C_{n-1,d}(\mathbb{P}^n)] t^d = \sum_d \frac{1 - [\mathbb{A}^1]^{\binom{n+d}{d}}}{1 - [\mathbb{A}^1]} t^d \tag{1}$$

is irrational. However, one may notice that in the irrational power series above, if one takes the limit $[\mathbb{A}^1] \to 1$, then we can recover the rationality of the Euler-Chow series. So, we can reformulate our last question as what happens if we change the coefficients in the above series, in other words, what happens if one considers the above formal power series (1) with coefficients the Chow motives of Chow varieties, modulo the relation $[\mathbb{A}^1] = 1$. Is it rational? If it is, what geometric or algebraic information can be read from it?

The goal of this paper is to answer these questions as much as possible. We are just going to get a glimpse of something that may morph to a deep subject, with a very hard and interesting questions.

Let us consider the K-ring $K(\mathcal{C}hM)$ of Chow motives, modulo $\mathbb{A}^1$ homotopy, namely we identify $[X \times \mathbb{A}^1]$ with $[X]$. Now, define the Motivic Chow series by $MC_p(X) := \sum [C_{p,d}(X)]t^d$ with coefficients in $K(\mathcal{C}hM)$. We will show this series is rational in many cases.

(1) Picard Product formula, see Thm 3.10 . When $\mathrm{Pic}(X) \times \mathrm{Pic}(X) = \mathrm{Pic}(X \times Y)$, then we have $MC_{n-1}(X)MC_{m-1}(Y) = MC_{n+m-1}(X \times Y)$, where $n = \dim X$ and $m = \dim Y$. In particular, when X and Y are curves, then $MC_0(X)$ and $MC_0(Y)$ are always rational by the result of Kapranov, hence for very general curves X and Y, $MC_1(X \times Y)$ is rational.

(2) Torus action, see Thm 4.2: When the multiplicative group $\mathbb{G}_m$ acts on X, then, roughly speaking, X is the disjoint union of the fixed locus $X^{\mathbb{G}_m}$ and the free orbits X_F. If the quotient $\overline{X} = X_F/\mathbb{G}_m$ exists, then the free orbit part looks like $\overline{X} \times \mathbb{G}_m$, and its class in $K(\mathcal{C}hM)_{\mathbb{A}^1}$ is zero because $[\overline{X} \times \mathbb{G}_m] = [\overline{X} \times \mathbb{A}^1] - [\overline{X}]$. Thanks to Thomason's Torus generic slice theorem, see [16, Prop. 4.10], one can justify this rough argument to show that $[X] = [X^{\mathbb{G}_m}]$. Moreover, it works for more general torus action to get $[X] = [X^T]$ when the algebraic torus T acts on X. This is a powerful tool to compute the Motivic Chow series. For example, one can show that the Motivic Chow series of toric variety is always rational.

One important feature of the Motivic Chow series is that it can detect very subtle geometric property of varieties. For example, when one blows up $\mathbb{P}^2$ along 3 points, then one can compute its cohomology group, Chow group, higher Chow group, algebraic cobordims , or even Lawson cohomology group without knowing the configuration of the 3 points in $\mathbb{P}^2$. But if one needs to calculate its Motivic Chow series, the answer depends on if the 3 points are colinear or not, see examples 4.5 and 4.6. We should observe that example 4.6 is new. It has not been able to be computed yet with the Euler-Chow series.

We don't know if there exists any cohomology theory whose odd part and even part correspond to the numerator and the denominator of the Motivic Chow series, but if such a cohomology theory exists, then it will be a powerful tool to study algebraic varieties.

Convention: Unless explicitly said, throughout this paper we work in the category of algebraic varieties over κ, a closed field of characteristic zero. We also denote by R any of the K-rings used in this paper, for details see Remark 1.10.

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1. K-rings of categories

Definition 1.1. *Let $\mathcal{C}$ be a category which has "addition" (say $X \amalg Y$) and "multiplication" (say $X \times Y$) such that*

(1) *Addition is commutative and associative, namely $X \amalg Y \simeq Y \amalg X$, and $(X \amalg Y) \amalg Z \simeq X \amalg (Y \amalg Z)$.*
(2) *Multiplication is commutative and associative, namely $X \times Y \simeq Y \times X$, and $(X \times Y) \times Z \simeq X \times (Y \times Z)$.*
(3) *Distributive law holds, namely $X \times (Y \amalg Z) \simeq (X \times Y) \amalg (X \times Z)$.*

An object $\mathbf{1} \in \mathcal{C}$ is called the multiplicative unit *if $\mathbf{1} \times X \simeq X$ for any $X \in \mathcal{C}$. We always assume that the multiplicative unit exists. Then we define* the K-ring of $\mathcal{C}$, *denoted as $K(\mathcal{C})$, to be the ring, generated by the objects of $\mathcal{C}$ (we write the class of X to be $[X]$), under the relations*

(i) *if X and Y are isomorphic, then $[X] = [Y]$.*
(ii) $[X \amalg Y] - [X] - [Y] = 0$.
(iii) $[X \times Y] - [X] \cdot [Y] = 0$.
(iv) $[\mathbf{1}] - 1 = 0$.

Sometimes, we consider the K-ring modulo extra relations, and we denote the quotient as $K'(\mathcal{C})$ when the extra relation is clear.

Remark 1.2. When $\mathcal{C} = \mathcal{F}Sets$ is the category of finite sets, usual disjoint union and multiplication satisfies the conditions, and $K(\mathcal{F}Sets) \simeq \mathbb{Z}$, by sending $[X]$ to $|X|$.

When $\mathcal{C} = \mathcal{T}op$ is the category of "good" topological spaces (e.g., finite simplicial complexes), and if we add the extra relation $[X] = [C] + [U]$ where $U \subset X$ is an open subset and $C \subset X$ the complement of U, then $K'(\mathcal{T}op) \simeq \mathbb{Z}$, by sending $[X]$ to $\chi(X)$, the Euler characteristic of X (see [18]).

When a multiplicative unit $\mathbf{1}$ exists, then its isomorphism class is unique. If $\mathbf{1}'$ is another multiplicative unit, then $\mathbf{1} \simeq \mathbf{1} \times \mathbf{1}' \simeq \mathbf{1}'$.

When $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves addition, multiplication, and the multiplicative unit if it exists, then F induces a ring homomorphism $K(\mathcal{C}) \to K(\mathcal{D})$.

Definition 1.3. *Let $\mathcal{C} = \mathcal{V}ar/\kappa$ be the category of algebraic varieties over κ. We define the addition in $\mathcal{V}ar/\kappa$ to be the disjoint union and the multiplication to be the product. We also consider the extra relation $[X] = [C] + [U]$ where $U \subset X$ is an open subscheme and $C \subset X$ the compliment of U with the reduced scheme structure. In this paper, $K'(\mathcal{V}ar/\kappa)$ always means $K(\mathcal{V}ar/\kappa)$ modulo this extra relation.*

Let $\mathcal{C}hM$ be the category of pure Chow motives. Its objects are triple (X, p, n) where X is a smooth projective variety over κ, $p : X \vdash X$ an idempotent correspondence, and $n \in \mathbb{Z}$ (see [15]). The category of Chow motives can be considered as "the universal cohomology theory".

In $\mathcal{C}hM$, we define addition as the direct sum and the multiplication as tensor product.

Theorem 1.4. *(Bittner [2]) $K'(\mathcal{A}Var)$ is canonically isomorphic to the ring generated by smooth projective varieties, modulo the relation $[X]+[E] = [C]+[Bl_C X]$ where $C \subset X$ is a smooth closed subvariety, $Bl_C X \to X$ the blowing up along C, and $E \subset Bl_C X$ the exceptional divisor.*

Corollary 1.5. *There is a ring homomorphism $K'(\mathcal{V}ar/\kappa) \to K(\mathcal{C}hM)$ which sends $[X]$ to $[(X, \Delta_X, 0)]$.*

Proof. It follows from the fact that in $\mathcal{C}hM$, we have $ch(X) \oplus ch(E) \simeq ch(C) \oplus ch(Bl_C X)$. □

Remark 1.6. Using this ring homomorphism $K'(\mathcal{V}ar/\kappa) \to K(\mathcal{C}hM)$, we can define $ch(X)$ to be the image of $[X]$, even when X is not projective nor smooth. For example, $ch(\mathbb{A}^1) = ch(\mathbb{P}^1) - ch(\mathrm{Pt})$, and when X is a nodal rational curve, then $ch(X) = ch(\mathbb{A}^1)$.

The following lemma will be useful in the next section.

Lemma 1.7. *Let $f : X \to Y$ be a proper morphism of reduced schemes, which is bijective set theoretically. Then $[X] = [Y]$ in $K'(\mathcal{V}ar/\kappa)$.*

In this case, as a morphism of topological spaces, f is a homeomorphism.

Proof. First, we show that f is a homeomorphism. As f_{set} is continuous, we need to show that f_{set}^{-1} is also continuous. As f is proper, the image of a closed subset of X by f_{set} is closed, hence f_{set}^{-1} is continuous.

We may decompose X and Y into locally closed irreducible subschemes, so we may assume that X and Y are varieties. Let $\eta \in X$ be the generic point, then as f is a homeomorphism, $f(\eta) \in Y$ is also the generic point. The extension degree of $K(\eta)/K(f(\eta))$ must be 1, otherwise f cannot be bijective in characteristic 0. Then f is birational, and there is an open subscheme $U \subset Y$ such that $f^{-1}(U) \to U$ is isomorphic, and in particular, $[f^{-1}(U)] = [U]$ in $K'(\mathcal{V}ar/\kappa)$. We need to show that $[X - f^{-1}(U)] = [Y - U]$, but $f_{|X-f^{-1}(U)} : X - f^{-1}(U) \to Y - U$ can be regarded as a proper morphism of reduced schemes, which is bijective set theoretically. Hence by Noetherian induction, we are reduced to the 0-dimensional case, where Lemma is obvious. □

Example 1.8. When X is a cuspidal rational curve, then its normalization is a proper morphism which is bijective set theoretically, and $[X] = [\mathbb{P}^1]$ in $K'(\mathcal{V}ar/\kappa)$.

Definition 1.9. *When $\mathcal{C} = \mathcal{H}S$ is the category of the pure Hodge structures with addition direct sum and multiplication tensor product, then we can define $K(\mathcal{H}S)$. For each pure Chow motive M, its cohomology group $H^*(M)$ is well-defined and it has a Hodge structure, which defines a ring homomorphism $K(\mathcal{C}hM) \to K(\mathcal{H}S)$.*

When $\mathcal{C} = \mathcal{V}ec^{\pm}$ is the category of $\mathbb{Z}_2$-graded vector spaces, with objects $V = V^{even} \oplus V^{odd}$. We define the addition as the direct sum and multiplication as the tensor product, with $\mathbb{Z}_2$-grading following the usual convention.

Then we can find $K(\mathcal{V}ec^{\pm}) \simeq \mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$*, where* $[V^{even} \oplus V^{odd}]$ *is sent to* $\dim(V^{even}) + \epsilon \dim(V^{odd})$*. There is a forgetful functor from* $\mathcal{H}S$ *to* $\mathcal{V}ec^{\pm}$*, which induces a natural ring homomorphism* $K(\mathcal{H}S) \to K(\mathcal{V}ec^{\pm})$*.*

From $K(\mathcal{V}ec^{\pm}) \simeq \mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$*, one can define a ring homomorphism to* $\mathbb{Z}$ *by substituting* $\epsilon = -1$*. The composition* $K'(\mathcal{V}ar/\kappa) \to K(\mathcal{C}hM) \to K(\mathcal{H}S) \to K(\mathcal{V}ec^{\pm}) \to \mathbb{Z}$ *sends* $[X]$ *to* $\chi(X)$*, the Euler characteristic.*

For $\mathcal{C} = \mathcal{V}ar/\kappa$*, we can add one more relation, namely* $[X \times \mathbb{A}^1] = [X]$*, which is called* $\mathbb{A}^1$*-homotopy relation. We denote* $K'(\mathcal{V}ar/\kappa)$ *modulo* $\mathbb{A}^1$*-homotopy relation as* $K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$*. Similarly, for* $\mathcal{C} = \mathcal{C}hM$*, we can also consider* $\mathbb{A}^1$*-homotopy relation, and we write* $K(\mathcal{C}hM)$ *modulo* $\mathbb{A}^1$*-homotopy relation as* $K(\mathcal{C}hM)_{\mathbb{A}^1}$*. This is same as ignoring the dimension shifting, namely* $[(X, p, n)] \sim [(X, p, m)]$ *for any* $n, m \in \mathbb{Z}$*.*

$\mathbb{A}^1$*-homotopy relation for the Hodge structure is equivalent to* $[V(1)] = [V]$*, ignoring the Tate twists. We write* $K(\mathcal{H}S)$ *ignoring the Tate twists as* $K(\mathcal{H}S)_{Tate}$*.*

Summarizing, we have the following commutative diagram for K*-rings.*

$$\begin{tikzcd}
K'(\mathcal{V}ar/\kappa) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K(\mathcal{C}hM) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K(\mathcal{H}S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K(\mathcal{V}ec^{\pm}) \arrow[r] & \mathbb{Z} \\
K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1} \arrow[r] & K(\mathcal{C}hM)_{\mathbb{A}^1} \arrow[r] & K(\mathcal{H}S)_{Tate} \arrow[ur] & &
\end{tikzcd}$$

Remark 1.10. When some argument works in any of the rings in the above diagram, we simply represent them by R.

2. Chow varieties

Definition 2.1. *Let* $X \subset \mathbb{P}^n$ *be a projective variety, then we denote by* $C_{p,d}(X)$ *the Chow variety of* X *which parametrizes effective* p*-cycles of degree* d *in* X*, see* [3]*. We consider* $C_{p,d}(X)$ *with its natural reduced scheme structure. We write* $C_p(X) := \amalg_{d \geq 0} C_{p,d}(X)$*, and* $B_p(X)$ *the set of connected components of* $C_p(X)$*.*

Remark 2.2. There is one-to-one correspondence between the closed points of $C_{p,d}(X)$ and the effective cycles $\sum a_i V_i$ of degree d and dimension p. Hence the closed points of $C_p(X)$ corresponds one-to-one to effective p cycles on X, and it is independent of the choice of the embedding $X \subset \mathbb{P}^n$ set theoretically, in fact, as a topological space with its Zariski topology, see [9], but Nagata showed that the scheme structure depends on the embedding [14]. When $C_p(X \overset{\varphi}{\hookrightarrow} \mathbb{P}^n)$ and $C_p(X \overset{\psi}{\hookrightarrow} \mathbb{P}^m)$ are two Chow varieties coming from different embeddings into projective spaces, then using the Chow variety $\widetilde{C}$ for the embedding $X \overset{\varphi \times \psi}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ where $N = nm + n + m$ with $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ the Segre embedding, Hoyt proved that there are morphisms $\widetilde{C}$ to $C_p(X \overset{\varphi}{\hookrightarrow} \mathbb{P}^n)$ and $C_p(X \overset{\psi}{\hookrightarrow} \mathbb{P}^m)$ which are

proper bijective [9]. Hence by Lemma 1.7, we have

$$[C_p(X \overset{\varphi}{\hookrightarrow} \mathbb{P}^n)] = [C_p(X \overset{\psi}{\hookrightarrow} \mathbb{P}^m)] \text{ in } K'(\mathcal{V}ar/\kappa)$$

hence the class $[C_p(X)] \in K'(\mathcal{V}ar/\kappa)$ is independent of the choice of the embedding into projective spaces.

Remark 2.3. When X is projective, the Chow variety $C_{p,d}(X)$ is also projective. It is constructed as a closed subscheme of $C_{p,d}(\mathbb{P}^n)$, and when $Y \subset X$ is a closed subscheme, then $C_{p,d}(Y)$ is a closed subscheme of $C_{p,d}(X)$. The addition of effective cycles determines a morphism of schemes $C_{p,d}(X)\times C_{p,e}(X) \to C_{p,d+e}(X)$, which gives an abelian monoid object structure on $C_p(X)$, see Friedlander [7, Prop. 1.8]. In particular, its connected components $B_p(X)$ have an abelian monoid structure. The group completion of the monoid $B_p(X)$ is $\mathrm{CH}_p(X)/\overset{alg}{\sim}$, the Chow group modulo algebraic equivalence [7, Prop. 1.8].

Definition 2.4. *Let X be a quasi-projective reduced scheme, and $\tilde{X} \supset X$ be a projective completion, namely $\tilde{X}$ is a projective scheme which has X as an open subscheme. Let $Y \subset \tilde{X}$ be the complement of X with reduced scheme structure, then $C_p(Y)$ is a closed subscheme of $C_p(\tilde{X})$. The zero degree part of $C_p(Y)$, namely $C_{p,0}(Y)$ consists of one point corresponding to ϕ, and we define $C_p(Y)^+ := \amalg_{d>0} C_{p,d}(Y)$, which is again a closed subscheme of $C_p(\tilde{X})$. We define* the Chow variety of the quasi-projective variety X in $\tilde{X}$ *to be the complement of the image of $C_p(Y)^+ \times C_p(\tilde{X}) \to C_p(\tilde{X})$ by the addition morphism in $C_p(\tilde{X})$, denoted as $C_p(X \subset \tilde{X})$, which depends on the embedding $\tilde{X} \in \mathbb{P}^n$. In other words, $C_p(X \subset \tilde{X})$ is an open subscheme of $C_p(\tilde{X})$ which consists of cycles $\sum a_i[V_i]$ with $V_i \cap X \neq \phi$.*

More generally, when X is a locally closed subscheme of a projective scheme Y, then take $\tilde{X}$ to be any closed subscheme of Y which contains X. Then $C_p(X \subset \tilde{X})$ consists of subset of $C_p(Y)$ which consists of cycles $\sum a_i[V_i]$ with $V_i \cap X \neq \phi$, hence it is independent of the choice of $\tilde{X}$, so we define $C_p(X \subset Y)$ to be $C_p(X \subset \tilde{X})$ for any such $\tilde{X}$.

Remark 2.5. By Remark 2.2, the class $[C_p(X \subset Y)] \in K'(\mathcal{V}ar/\kappa)$ is independent of the embedding $\tilde{X} \subset \mathbb{P}^n$.

Set theoretically the closed points of $C_p(X \subset Y)$ correspond one-to-one to the effective p-cycles on X, and hence it is independent of the choice of embedding $X \subset \tilde{X}$.

But as a topological space, $C_p(X \subset Y)$ does depend on the embedding. For example, let $P \in \mathbb{P}^2$ be a point, $X = \mathbb{P}^2 - P$, $Y_1 = \mathbb{P}^2$ and Y_2 the blow-up of Y_1 along P. Then any two lines in $C_1(X \subset Y_1)$ are in the same connected component, but in $C_1(X \subset Y_2)$, the lines whose closure is through P are in a different connected component as the other lines.

Lemma 2.6. *Let X be a locally closed subscheme of a projective scheme Y, and $f : X \to Z$ a flat morphism of relative dimension k to a quasi-projective*

scheme Z. Let $\tilde{Z}$ be any projective completion of Z, and $T \subset C_p(Z \subset \tilde{Z})$ be any reduced closed subscheme. Then there exists a stratification $T = \amalg T_i$ into locally closed subschemes of T together with morphisms $\varphi_i : T_i \to C_{p+k}(X \subset Y)$ such that for any point $t \in T_i$ corresponding to a cycle $t = \sum n_i V_i$, the image of the morphism $\varphi_i(t)$ corresponds to $f^*(t)$, the flat pull-back by f (see Fulton [8, Chap. 1]*).*

Moreover, in each connected component $C_{p,\alpha}(X \subset Y)$, the class $\sum_i [\varphi_i(T_i) \cap C_{p+k,\alpha}(X \subset Y)] \in K'(\mathcal{V}ar/\kappa)$ is independent of the choice of the projective completion $Z \subset \tilde{Z}$.

Proof. By Noetherian induction, it is enough to show that there is a non-empty open subscheme $U \subset T$ with a morphism $\varphi : U \to C_p(X \subset Y)$ such that for any $u \in U$, $\varphi(u)$ corresponds to $f^*(u)$ as a cycle.

If $\mathrm{Im}(f) \cap T$ is not dense in T, then we can take $U \subset T$ so that U does not intersect with the image of f, and take the constant morphism $U \to \{\varphi\} \in C_p(X \subset Y)$, so we may assume that $\mathrm{Im}(f) \cap T$ is dense in T. As flat morphism is open, by replacing T with $\mathrm{Im}(f) \cap T$, we may assume that $\mathrm{Im}(f) \supset T$.

Let $U_1 \subset T$ be an irreducible component of the smooth locus of the reduced scheme T. By [7, Theorem 1.4], the embedding $U_1 \to C_p(Z \subset \tilde{Z})$ corresponds to a cycle $\sum n_i W_i$ of $U_1 \times \tilde{Z}$ such that for any point $u \in U$, the cycle $u \in C_p(Z \subset \tilde{Z})$ corresponds to $\sum n_i [W_i \cap (\{u\} \times Z)]$. In particular the image of W_i contains a dense open subscheme of U_1.

We pull-back the cycle $\sum n_i [W_i \cap (U_1 \times Z)]$ by the flat morphism $1_{U_1} \times f$ to obtain $(1_{U_1} \times f)^* \sum n_i [W_i \cap (U_1 \times Z)] = \sum m_j [\tilde{W}_j]$. Let $\overline{W_j}$ be the closure of $\widetilde{W}_j$ in $U_1 \times Y$. As the fiber of $\overline{W_j} \to U_1$ at the generic point is irreducible, it is also irreducible in some open subscheme $U \subset U_1$ for all j, hence for $u \in U$, we have $\overline{W_j} \cap \{u\} \times Y = \overline{\widetilde{W}_j \cap \{u\} \times Y}$. Again by [7, Theorem 1.4], there is a morphism $\varphi : U \to C_{p+k}(Y)$ such that for $u \in U$, the point $\varphi(u)$ corresponds to the cycle $\sum m_j [\overline{W_j} \cap \{u\} \times Y]$, which corresponds to $f^*(u)$, and in particular the image is in $C_{p+k}(X \subset Y)$. We have constructed $\varphi : U \to C_{p+k}(X \subset Y)$.

For $t \in T$, say $t \in T_i$, then $\varphi_i(t)$ corresponds to $f^*(t) \in C_{p+k}(Y)$, which is independent of the choice of the projective completion $Z \subset \tilde{Z}$, and in particular the union of the images of φ_i is independent of the choice of the projective completion, from which it follows that $\sum_i [\varphi_i(T_i) \cap C_{p,\alpha}(X \subset Y)] \in K'(\mathcal{V}ar/\kappa)$ is independent of the choice of $Z \subset \tilde{Z}$. $\square$

Definition 2.7. *Let X be a locally closed subscheme of a projective variety Y, and $f : X \to Z$ be a flat morphism of relative dimension k, and $Z \subset \tilde{Z}$ a projective completion. For a subvariety $V \subset C_p(Z \subset \tilde{Z})$ and $\alpha \in B_{p+k}(Y)$, we define* the α-component of $f^*[V]$ *to be*

$$f^*[V]_\alpha := \sum_i [\varphi_i(V_i) \cap C_{p+k,\alpha}(X \subset Y)] \in K'(\mathcal{V}ar/\kappa)$$

where $V = \amalg V_i$ *is a stratification into locally closed subschemes of* V*, and* $\varphi_i : V_i \to C_{p+k}(X \subset Y)$ *the morphism which maps* $v \in V_i$ *to the point corresponding to the cycle* $f^*(v)$*, as in Lemma 2.6.*

For a linear combination of subvarieties, we extend f^* *linearly. By Lemma 2.6,* $f^*[V]_\alpha$ *is independent of the choice of the projective completion* $Z \subset \tilde{Z}$*.*

3. Motivic Chow Series

Definition 3.1. *An additive monoid* S *is called to have* finite fiber *when for any* $s \in S$*, the set* $\{(a,b) \in S \times S | a + b = s\}$ *is a finite set. We assume that additive monoid* S *always have an additive identity* 0*. When* R *is a commutative ring with a muliplicative unit* 1*, and* S *is an additive monoid with finite fiber, we define* the formal power series over S with coefficient in R *to be the set of functions from* S *to* R*, written as* $R[[S]]$*, and we write an element of* $R[[S]]$ *as* $f = \sum_{s \in S} a_s t^s$*, where* f *sends* s *to* a_s*. Define the addition in* $R[[S]]$ *as the usual addition as functions;* $(\sum a_s t^s) + (\sum b_s t^s) = \sum (a_s + b_s) t^s$*. We define the multiplication in* $R[[S]]$ *by convolution;* $(\sum a_s t^s) \cdot (\sum b_s t^s) := \sum_{s \in S} \left(\sum_{s_1 + s_2 = s} a_{s_1} b_{s_2} \right) t^s$*, where the sum is finite, because* S *has finite fiber.*

A power series $f = \sum a_s t^s \in R[[S]]$ *is called* a polynomial *when* $a_s = 0$ *except for finitely many* $s \in S$*. We denote the subring of all polynomials as* $R[S]$*. A polynomial* $f = \sum a_s t^s \in R[S]$ *is called* monic *when its coefficient of* t^0 *is* 1 *(namely as a function,* f *sends* 0 *to* 1*). A power series* $f \in R[[S]]$ *is called* a rational function *if there exists a monic polynomial* $g \in R[S]$ *such that* $fg \in R[[S]]$ *is a polynomial.*

Remark 3.2. When X is a projective variety, the monoid of connected components of the Chow variety $B_p(X)$ (see Definition 2.1 and Remark 2.3) has finite fiber, because for each fixed degree d, the components of the Chow variety $C_p(X)$ corresponding to the cycles with degree less than or equal to d is a finite set. When $S_1 \to S_2$ is a monoid homomorphism, then we have a ring homomorphism $R[[S_1]] \to R[[S_2]]$. For example, when Y is a closed subscheme of a projective variety X, then the monoid homomorphism $B_p(Y) \to B_p(X)$ induces $R[[B_p(Y)]] \to R[[B_p(X)]]$. A ring homomorphism $R_1 \to R_2$ canonically induces a ring homomorphism $R_1[[S]] \to R_2[[S]]$.

Definition 3.3. *When* X *is a projective variety, then we define its* Motivic Chow Series of dimension p *to be*

$$MC_p(X) := \sum_{\alpha \in B_p(X)} [C_{p,\alpha}(X)] t^\alpha,$$

in $R[[B_p(X)]]$*. For* R *see Remark 1.10.*

When X *is a quasi-projective variety with a fixed embedding into a projective variety* $X \subset \tilde{X}$*, we define its* Motivic Chow Series of dimension p in $\tilde{X}$ *to be*

$$MC_p(X \subset \tilde{X}) := \sum_{\alpha \in B_p(\tilde{X})} [C_{p,\alpha}(X \subset \tilde{X})] t^\alpha.$$

Example 3.4. When $p = 0$, the degree d Chow variety $C_{0,d}(X)$ is in the same class as $Sym^d(X)$ in $K'(\mathcal{V}ar/\kappa)$, hence the Motivic Chow series of $p = 0$ is the Motivic zeta. Kapranov [10] proved that when X is a smooth projective curve, then $MC_0(X)$ is rational in $K'(\mathcal{V}ar/\kappa)[[B_0(X)]] \simeq K'(\mathcal{V}ar/\kappa)[[t]]$, in the ring of formal power series with one variable t. When X is a surface, Larsen and Lunts [12] proved that $MC_0(X) \in K'(\mathcal{V}ar/\kappa)[[B_0(X)]] \simeq K'(\mathcal{V}ar/\kappa)[[t]]$ is rational if and only if X is a ruled surface. On the other hand, using the notion of finite dimensionality of motives, if the Chow motive of X is finite dimensional, then $MC_0(X) \in K(\mathcal{C}hM)[[B_0(X)]] \simeq K(\mathcal{C}hM)[[t]]$ is rational [1]. According to Bloch-Beilinson Conjecture, all motives should be finite dimensional, and when X is a product of curves or Abelian variety, their motives are finite dimensional, hence $MC_0(X) \in K(\mathcal{C}hM)[[B_0(X)]] \simeq K(\mathcal{C}hM)[[t]]$ is rational [11]. When $X = \mathbb{P}^n$ with $n \geq 2$, then $MC_{n-1}(X) \in K(\mathcal{C}hM)[[B_{n-1}]]$ is not rational by [6]. When X is a toric variety, then $\displaystyle\sum_{\alpha \in H_2p(X,\mathbb{Z})} \chi(C_{p,\alpha}(X))t^\alpha \in \mathbb{Z}[[H_{2p}(X,\mathbb{Z})]]$ is rational by [5]. Notice that there is a ring homomorphism $K(\mathcal{C}hM)_{\mathbb{A}^1} \to \mathbb{Z}$ which sends $ch(X)$ to $\chi(X)$. Macdonald's theorem says that $\sum \chi(C_{0,d}(X))t^d = \dfrac{1}{(1-t)^{\chi(X)}}$.

Theorem 3.5. *(Localization) Let p be an non-negative integer, X a projective variety, $U \subset X$ an open subvariety and $Y \subset X$ the complement of U in X with the reduced closed subscheme structure. Let $\overline{MC_p(Y)}$ be the image of $MC_p(Y)$ by the ring homomorphism $R[[B_p(Y)]] \to R[[B_p(X)]]$ (see Remark 3.2). Then in $R[[B_p(X)]]$, we have*

$$\overline{MC_p(Y)} \cdot MC_p(U \subset X) = MC_p(X).$$

Proof. Because there are ring homomorphisms from $K'(\mathcal{V}ar/\kappa)$ to all other K-rings, it is enough to show in the case $R = K'(\mathcal{V}ar/\kappa)$.

Let $\varphi : B_p(Y) \to B_p(X)$ be the natural morphism of monoids. For each $\overline{\alpha} \in B_p(X)$, we define $C_{p,\overline{\alpha}}(Y) := \displaystyle\bigcup_{\varphi(\alpha)=\overline{\alpha}} C_{p,\alpha}(Y)$. Set theoretically, each connected component $C_{p,\gamma}(X)$ of the Chow variety of X is a disjoint union of the images $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X)$ by the addition morphism, with the index set $\{(\overline{\alpha}, \beta) \in B_p(X) \times B_p(X) | \overline{\alpha} + \beta = \gamma\}$. We denote the image of $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X)$ by $W_{\overline{\alpha},\beta} \subset C_{p,\gamma}(X)$. We will show that $W_{\overline{\alpha},\beta}$ is a locally closed subscheme of $C_{p,\gamma}(X)$, and the restriction of the addition morphism $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X) \to W_{\overline{\alpha},\beta}$ is a proper bijection. Because each irreducible cycle V in X is contained in one, and only one of $C_p(Y)$ and $C_p(U \subset X)$, the way to decompose a cycle in $C_p(X)$ into the sum of $C_p(Y)$ and $C_p(U \subset X)$ is unique, from which the bijectivity follows.

As $C_{p,\overline{\alpha}}(Y) \subset C_p(X)$ is closed, the image of $C_{p,\overline{\alpha}} \times C_{p,\beta}(X)$ in $C_{p,\gamma}(X)$ is closed. As $C_{p,\beta}(U \subset X) \subset C_{p,\beta}(X)$ is open, the image of $C_{p,\overline{\alpha}}(Y) \times$

$(C_{p,\beta}(X) - C_{p,\beta}(U \subset X))$ in $C_{p,\gamma}(X)$ by the addition morphism is also closed. By the following Lemma 3.6, if you remove the image of $C_{p,\overline{\alpha}}(Y) \times (C_{p,\beta}(X) - C_{p,\beta}(U \subset X))$ in $C_{p,\gamma}(X)$ from the image of $C_{p,\overline{\alpha}} \times C_{p,\beta}(X)$, then the remaining subset is exactly $W_{\overline{\alpha},\beta}$, hence $W_{\overline{\alpha},\beta}$ is locally closed.

Lemma 3.6. *The inverse image of $W_{\overline{\alpha},\beta}$ by the addition morphism $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(X) \to C_{p,\gamma}(X)$ is $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X)$.*

Proof. (of Lemma 3.6) Each cycle corresponding to a point in $W_{\overline{\alpha},\beta}$ can be written as $c + d$ with $c \in C_{p,\overline{\alpha}}(Y)$ and $d \in C_{p,\beta}(U \subset X)$. Assume that it has another expression $c + d = c' + d'$ with $c' \in C_{p,\overline{\alpha}}(Y)$ and $d' \in C_{p,\beta}(X)$. As $C_{p,\overline{\alpha}}(Y) \subset C_{p,\overline{\alpha}}(X)$ and $C_{p,\overline{\alpha}}(X)$ is a connected component, we have $\deg c = \deg c'$. c contains all the Y-supported cycles in $c + d = c' + d'$ with c' supported on Y, the cycle $c - c'$ is effective and degree 0, hence $c = c'$, therefore $d = d'$, and $(c', d') \in C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X)$. (end of the proof of Lemma 3.6.) $\square$

From Lemma 3.6, it follows that the morphism $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X) \to W_{\overline{\alpha},\beta}$ is the base extension of the proper morphism $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(X) \to C_{p,\gamma}(X)$ by the inclusion $W_{\overline{\alpha},\beta} \to C_{p,\gamma}(X)$, hence $C_{p,\overline{\alpha}}(Y) \times C_{p,\beta}(U \subset X) \to W_{\overline{\alpha},\beta}$ is proper. By Lemma 1.7, it follows that $[C_{p,\overline{\alpha}}(Y)] \times [C_{p,\beta}(U \subset X)] = [W_{\overline{\alpha},\beta}]$.

Set theoretically $C_{p,\gamma}(X)$ is a disjoint union of locally closed subschemes $W_{\overline{\alpha},\beta}$, in $K'(\mathcal{V}ar/\kappa)$, hence in $K(\mathcal{C}hM)$, and in $K(\mathcal{C}hM)_{\mathbb{A}^1}$. Therefore $[C_{p,\gamma}(X)] = \sum_{\overline{\alpha}+\beta=\gamma} [C_{p,\overline{\alpha}}(X)] \times [C_{p,\beta}(U \subset X)]$. Now we have

$$\begin{aligned}
&\overline{MC_p(Y)} \times MC_p(U \subset X) = \\
&\quad = \left(\sum_{\overline{\alpha} \in B_p(X)} [C_{p,\overline{\alpha}}(Y)] t^{\overline{\alpha}} \right) \times \left(\sum_{\beta \in B_p(X)} [C_{p,\beta}(U \subset X)] t^{\beta} \right) \\
&\quad = \sum_{\gamma \in B_p(X)} \left(\sum_{\overline{\alpha}+\beta=\gamma} [C_{p,\overline{\alpha}}(Y)] \times [C_{p,\beta}(U \subset X)] \right) t^{\gamma} \\
&\quad = \sum_{\gamma \in B_p(X)} [C_{p,\gamma}(X)] t^{\gamma} \\
&\quad = MC_p(X).
\end{aligned}$$

$\square$

Corollary 3.7. *Assume that X is a projective variety, $Y \subset X$ is a closed subscheme and $U \subset X$ its complement. If two of following series $MC_p(Y)$, $MC_p(X)$ and $MC_p(U \subset X)$ are rational, then the other one is also rational.*

Proof. It follows from the fact that all $MC_p(Y)$, $MC_p(X)$ and $MC_p(U \subset X)$ are monic. $\square$

Example 3.8. When $Y = \{P_1, \dots, P_r\} \subset X$ is a finite set of r elements in a connected scheme X, then $\overline{MC_0(Y)} = \dfrac{1}{(1-t)^r}$, and $MC_0((X-Y) \subset X) \in K(\mathcal{C}hM)[[t]]$ is rational if and only if the motive of X is finite dimensional. In particular, when U is any smooth curve with its smooth completion X, $MC_0(U \subset X) \in K'(\mathcal{V}ar/\kappa)[[t]]$ is rational.

Remark 3.9. It is interesting to see that when $Z = C \backslash \{P_1, \dots, P_{r+1}\}$ where C is a smooth projective curve of genus g and $r \geq 1$, then $Sym^n[Z] = 0$ in $K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$ if $n > 2g + r - 1$. In particular, $MC_0(Z \subset C)$ is a polynomial.

It is fun to see how it works concretely when $Z = \mathbb{P}^1 \backslash \{a_1, a_2, \dots, a_r, \infty\}$. We have isomorphism $\mathbb{A}^r \simeq Sym^r(\mathbb{A}^1)$ by identifying $(b_1, b_2, \dots, b_r) \in \mathbb{A}^r$ with the r roots of $f(x) = x^r + b_1 x^{r-1} + \cdots + b^{r-1}x + b_r$ in $Sym^r \mathbb{A}^1$. Then under this identification with the degree r monic polynomials, $Sym^r Z$ corresponds to $\{f(x) | f(a_1) \neq 0, f(a_2) \neq 0, \dots, f(a_r) \neq 0\}$. The linear map $\mathbb{A}^r \to \mathbb{A}^r$ sending $(b_1, b_2, \dots, b_r)$ to $(f(a_1), f(a_2), \dots, f(a_r))$ (where $f(x) = x^r + b_1 x^{r-1} + \cdots + b^{r-1}x + b_r$) is isomorphism by Vandermonde, and by this new coordinate, we can see that $[Sym^r Z] = [\mathbb{G}_m^r] = 0$ in $K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$.

Theorem 3.10. *(Picard Product Formula) (1) Let X and Y be smooth projective varieties such that the natural morphism gives the bijection $Pic(X) \times Pic(Y) \simeq Pic(X \times Y)$. Then we have*

$$MC_{n-1}(X) \cdot MC_{m-1}(Y) = MC_{n+m-1}(X \times Y) \in K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}[[B_{n+m-1}(X \times Y)]],$$

where by the natural morphism $Pic(X) \times Pic(Y) \simeq Pic(X \times Y)$, we identify $B_{n-1}(X) \otimes B_{m-1}(Y) \simeq B_{n+m-1}(X \times Y)$, with $n = \dim X$ and $m = \dim Y$.

(2) The assumption $Pic(X) \times Pic(Y) \simeq Pic(X \times Y)$ holds if and only if the morphism of varieties $Pic^\circ(X) \to Pic^\circ(Y)$ are only the constant morphisms.

Proof. (1) Let $n = \dim X, m = \dim Y$ and $\mathrm{Pic}^\circ(X)$ and $\mathrm{Pic}^\circ(Y)$ be the Picard varieties. Let $NS(X) \subset H^2(X, \mathbb{Z})$ be the Neron-Severi group, then the closed points of $NS(X) \times \mathrm{Pic}^\circ(X)$ corresponds one-to-one to the element of $\mathrm{Pic}(X)$, the Pircard group of X, and similarly for $\mathrm{Pic}(Y)$.

For each $\alpha \in NS(X)$, the image of $C_{n-1}(X)$ in $\alpha \times \mathrm{Pic}^\circ(X)$ is closed, and we denote it as $V_\alpha \subset \alpha \times \mathrm{Pic}^\circ(X)$. We have a stratification of V_α into locally closed subschemes $V_\alpha = \amalg V_{\alpha,i}$ such that the inverse image of $V_{\alpha,i}$ in $C_{n-1}(X)$ is $V_{\alpha,i} \times \mathbb{P}^{d_i}$. When D is a divisor whose class is in $V_{\alpha,i}$, then we have $\dim H^0(X, \mathcal{O}(D)) = d_i + 1$. Using these notations, we can write $MC_{n-1}(X) \in K(\mathcal{C}hM)_{\mathbb{A}^1}[[B_{n-1}(X)]]$ as

$$\begin{aligned} MC_{n-1}(X) &= \sum_\alpha \left(\sum_i [\mathbb{P}^{d_i}] \cdot [V_{\alpha,i}] \right) t^\alpha \\ &= \sum_\alpha \left(\sum_i (d_i + 1)[V_{\alpha,i}] \right) t^\alpha. \end{aligned}$$

We define $W_\beta = \amalg W_{\beta,j} \subset \beta \times \mathrm{Pic}^\circ(Y)$ with the inverse image of $W_{\beta,j}$ in $C_{m-1}(Y)$ be $W_{\beta,j} \times \mathbb{P}^{e_j}$, we can write

$$MC_{m-1}(Y) = \sum_\beta \left(\sum_j (e_j+1)[W_{\beta,j}] \right) t^\beta.$$

When $D_1 \in V_{\alpha,i}$ and $D_2 \in W_{\beta,j}$ are effective divisors, then $\pi_X^* D_1 + \pi_Y^* D_2$ is in $V_{\alpha,i} \times W_{\beta,j} \subset NS(X) \times \mathrm{Pic}^\circ(X) \times NS(Y) \times \mathrm{Pic}^\circ(Y)$, and as

$$H^0(X \times Y, \mathcal{O}(\pi_X^* D_1 + \pi_Y^* D_2)) = H^0(X, \mathcal{O}(D_1)) \otimes H^0(Y, \mathcal{O}(D_2)),$$

its inverse image in $C_{n+m-1}(X \times Y)$ is $\mathbb{P}^{(d_i+1)(e_j+1)-1}$, hence we have

$$\begin{aligned} &MC_{n+m-1}(X \times Y) = \\ &\quad = \sum_{\alpha,\beta} \left(\sum_{i,j} [\mathbb{P}^{d_i e_j + d_i e_j}] \cdot [V_{\alpha,i}] \cdot [W_{\beta,j}] t^{\alpha+\beta} \right) \\ &\quad = \sum_{\alpha,\beta} \left(\sum_{i,j} [(d_i+1)(e_j+1)] \cdot [V_{\alpha,i}] \cdot [W_{\beta,j}] t^{\alpha+\beta} \right) \\ &\quad = \left(\sum_\alpha \left(\sum_i (d_i+1)[V_{\alpha,i}] \right) \right) \cdot \left(\sum_\beta \left(\sum_j (e_j+1)[W_{\beta,j}] \right) \right) \\ &\quad = MC_{2n-1}(X) \cdot MC_{2m-1}(Y) \end{aligned}$$

(2) First we assume that there is no non-constant morphisms $\mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(Y)$ as varieties. By restricting the line bundle on $X \times Y$ to $\{x\} \times Y$ and $X \times \{y\}$, one can easily recover the preimage, so the natural morphism $\mathrm{Pic}(X) \times \mathrm{Pic}(Y) \to \mathrm{Pic}(X \times Y)$ is always injective, so we need to show that it is also surjective under the assumption.

Take a line bundle L on $X \times Y$. Let $x \in X$ be a point, and $L_Y := L_{|\{x\} \times Y}$ be the restriction of L, considered as a line bundle on Y, and let $M = (\pi_Y^* L_Y^{-1}) \otimes L$, then $M_{|\{x\} \times Y}$ is the trivial line bundle. By the universality of $\mathrm{Pic}^\circ(Y)$, there exists a morphism $X \to \mathrm{Pic}^\circ(Y)$ such that $M \simeq (1_X \times \varphi)^*(\mathcal{P}_Y) \otimes \pi_X^* L_X$ for some line bundle L_X on X, where $\mathcal{P}_Y$ is the Poincaré line bundle on $Y \times \mathrm{Pic}^\circ(Y)$. The morphism $\varphi : X \to \mathrm{Pic}^\circ(Y)$ factors through $\tilde{\varphi} : \mathrm{Alb}(X) \to \mathrm{Pic}^\circ(Y)$, and as $\mathrm{Pic}^\circ(X)$ is isogenous to $\mathrm{Alb}(X)$, there is a surjection $\mathrm{Pic}^\circ(X) \to \mathrm{Alb}(X)$. So we have $\mathrm{Pic}^\circ(X) \to \mathrm{Alb}(X) \to \mathrm{Pic}^\circ(Y)$, and by assumption, the composition $\mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(Y)$ is a constant morphism. Because $\mathrm{Pic}^\circ(X) \to \mathrm{Alb}(X)$ is surjective, the morphism $\tilde{\varphi}$ is a constant morphism, and hence $\varphi : X \to \mathrm{Pic}^\circ(Y)$ is a constant morphism. The image of $x \in X$ is 0, because $M_{|\{x\} \times Y}$ is trivial, hence φ is a constant morphism to 0, and $\varphi^{-1}\mathcal{P}_Y$ is the trivial line bundle. Therefore $M = \pi_X^* L_X$, and as $M = \pi_Y^* L_Y^{-1} \otimes L$, we conclude $L = \pi_X^* L_X \otimes \pi_Y^* L_Y$. We

have shown that the natural morphism $\mathrm{Pic}(X) \times \mathrm{Pic}(Y) \to \mathrm{Pic}(X \times Y)$ is surjective, as required.

Conversely, assume that $\psi : \mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(Y)$ is a non-constant morphism. Take an isogeny $\mathrm{Alb}(X) \to \mathrm{Pic}^\circ(X)$ and let $\tilde{\varphi} : X \to \mathrm{Alb}(X) \to \mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(Y)$ be the composition, then it is a non-constant morphism, and $(1_X \times \tilde{\psi})^* \mathcal{P}_Y$ cannot be written as $\pi_X^* L_X \times \pi_Y^* L_Y$, because for line bundle $\pi_X^* L_X \times \pi_Y^* L_Y$, its restriction to $\{x\} \times Y$ is L_Y for any $x \in X$, but the restriction of $(1_X \times \tilde{\psi})^* \mathcal{P}_Y$ to $\{x\} \times Y$ corresponds to $\tilde{\psi}(x)$, which is not constant. Hence we have found a line bundle which is not in the image of $\mathrm{Pic}(X) \times \mathrm{Pic}(Y)$. □

Corollary 3.11. *Let $C_1, \dots C_r$ be smooth projective curves such that the morphisms of abelian varieties between their Jacobian varieties $J(C_i)$'s are only zero morphisms. Then $MC_{r-1}(C_1 \times C_2 \times \cdots \times C_r) = \prod MC_0(C_i)$. In particular, it is rational in $K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}[[B_{r-1}(C_1 \times \cdots \times C_r)]]$*

Proposition-Definition 3.12. *Let X be a projective variety and $Y \subset X$ a locally closed subscheme. Let $\tilde{Y} \subset X$ be a closed subscheme such that $Y \subset \tilde{Y} \subset X$, and Y is an open subscheme of $\tilde{Y}$. We define $C_p(Y \subset X)$ to be $C_p(Y \subset \tilde{Y})$, considered as a locally closed subscheme of $C_p(X)$, and $MC_p(Y \subset X) \in R[[B_p(X)]]$ to be the canonical image of $MC_p(Y \subset \tilde{Y})$ in $R[[B_p(X)]]$. Then $C_p(Y \subset X)$ and $MC_p(Y \subset X)$ are independent of the choice of $\tilde{Y}$. When $X = \amalg X_i$ is the set theoretic decomposition of X into locally closed subschemes, then we have $\prod MC_p(X_i \subset X) = MC_p(X)$.*

Proof. $C_p(Y \subset \tilde{Y})$ is an open subscheme of $C_p(\tilde{Y})$, and as $C_p(\tilde{Y})$ is a closed subscheme of $C_p(X)$, so we may regard $C_p(Y \subset \tilde{Y})$ as the reduced locally closed subscheme of $C_p(X)$ whose points consist of linear combinations of irreducible varieties in X whose scheme theoretic generic point is contained in Y. We have found a description of $C_p(Y \subset X)$ without using $\tilde{Y}$, so it is independent of the choice of $\tilde{Y}$. Let $\{\alpha\}$ be the set of connected components of $C_p(Y \subset X)$, say $C_p(Y \subset X) = \amalg C_{p,\alpha}(Y \subset X)$ as schemes. Each $C_{p,\alpha}(Y \subset X)$ is contained in one connected component of $C_p(X)$, and for each $\beta \in B_p(X)$, we have

$$MC_p(Y \subset X) = \sum_{\beta \in B_p(X)} \left(\sum_{C_{p,\alpha}(Y \subset X) \subset C_{p,\beta}(X)} [C_{p,\alpha}(Y \subset X)] \right) t^\beta$$

hence $MC_p(Y \subset X)$ is independent of the choice of $\tilde{Y}$.

When $X = \amalg X_i$ is a decomposition of X into locally closed subschemes, then at least one of X_i is open, say X_0. Then by Theorem 3.5, letting $Y := X - X_0$, we have $\overline{MC_p(Y)} MC_p(X_0 \subset X) = MC_p(X)$. Now we can proceed by induction on the number of locally closed subschemes, and the observation that $\overline{MC_p(Y)} = MC_p(Y \subset X)$, together with $MC_p(X) = \prod MC_p(X_i \subset X)$. □

Definition 3.13. *Let X be a locally closed subscheme of a projective variety Y, and $f : X \to Z$ a flat morphism of relative dimension k, and $Z \subset \tilde{Z}$ a projective completion. Define $MC_p(f^*Z \to Y)$ to be*

$$MC_p(f^*Z \to Y) := \sum_{\alpha \in B_p(Y)} \left(\sum_{\beta \in B_{p-k}(\tilde{Z})} f^*[C_{p-k}(Z \subset \tilde{Z})]_\alpha \right) t^\alpha$$

(see Definition 2.7 for the notation $f^*[C_{p-k}(Z \subset \tilde{Z})]_\alpha$.)

Definition 3.14. *Let X be a locally closed subscheme of a projective variety Y, $f : X \to Z$ a flat morphism of relative dimension k, and $Z \subset \tilde{Z}$ the projective completion.*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & Y \\ Z \arrow[r, hook] & \tilde{Z} \end{tikzcd}$$

This diagram is called good configuration in dimension p *when f is surjective and there is a monoid homomorphism $\varphi : B_p(\tilde{Z}) \to B_{p+k}(Y)$ such that for any cycle $c \in C_{p,\alpha}(Z \in \tilde{Z})$, the flat pull-back $f^*(c)$ is in $C_{p+k,\varphi(\alpha)}(X \subset Y)$.*

Proposition 3.15. *Let X be a non-empty locally closed subscheme of a projective variety Y, $f : X \to Z$ a flat morphism of relative dimension k, and $Z \subset \tilde{Z}$ the projective completion. When $\tilde{Z}$ is connected, then this configuration is a good configuration in dimension 0 if and only if there exists $0 \neq \beta \in B_p(Y)$ such that for any point $P \in Z$, we have $[f^{-1}(P)] \in C_{k,\beta}(X, Y)$.*

When our diagram is a good configuration in dimension p with the monoid homomorphism $\varphi : B_p(\tilde{Z}) \to B_{p+k}(Y)$, then the following two conditions hold.

1. *Let $\varphi_* : R[[B_p(\tilde{Z})]] \to R[[B_{p+k}(Y)]]$ be the ring homomorphism induced by φ, with R being $K'(\mathcal{V}ar/\kappa)$, $K(\mathcal{C}hM)$ or $K(\mathcal{C}hM)_{\mathbb{A}^1}$, then we have*
$$\varphi_*(MC_p(Z \subset \tilde{Z})) = MC_{p+k}(f^*Z \to Y).$$
2. *When $MC_p(Z \subset \tilde{Z})$ is rational, then $MC_{p+k}(f^*Z \to Y)$ is also rational.*

Proof. When $\tilde{Z}$ is connected, we have $B_0(\tilde{Z}) \simeq \mathbb{Z}_{\geq 0}$ with $C_{0,d}\tilde{Z} \simeq Sym^d \tilde{Z}$, as topological spaces. Each point $P \in \tilde{Z}$ can be regarded as an element of $\tilde{Z} \simeq C_{0,1}(\tilde{Z})$, and $f^*[P] := [f^{-1}(P)]$ being in the same component $C_{k,\beta}(Y)$ is a necessary condition for our diagram to be a good configuration. Conversely, assume that $[f^{-1}(P)] \in C_{0,\beta}(X \subset Y)$ for all point $P \in Z$. As X is non-empty, at least for one $P \in Z$, $f^{-1}(P)$ is not an empty set, so β cannot be the connected component of the empty set. In particular, f is surjective. For any cycle $P_1 + \cdots + P_d \in C_{0,d}(Z \subset \tilde{Z})$, we have $f^*(P_1 + \cdots + P_d) =$

$\sum_{i=1}^d f^*(P_i) \in C_{k,d\beta}(X \subset \tilde{Y})$, so we can define $\varphi : B_0(\tilde{Z}) \to B_k(Y)$ to be $\varphi(d) := d\beta$ to make our diagram a good configuration.

When our diagram is a good configuration in dimension p, then by definition of $MC_{p+k}(f^*Z \to Y)$, we have

$$MC_{p+k}(f^*Z \to Y) = \sum_{\beta \in B_p(Y)} \left(\sum_{\alpha \in B_p(\tilde{Z})} f^*[C_p(Z \subset \tilde{Z})]_\beta \right) t^\beta$$

where

$$f^*[C_p(Z \subset \tilde{Z})]_\beta = \sum_i [\varphi_i(V_i) \cap C_{p+k,\beta}(X \subset Y)]$$

with $C_p(X \to Y) = \amalg V_i$ is a stratification into locally closed subschemes, $\varphi_i : V_i \to C_p(X \to Y)$ sends $c \in C_p(Z \subset \tilde{Z})$ to $f^*(c) \in C_p(X \subset Y)$. Because our diagram is a good configuration, for each $\alpha \in B_p(\tilde{Z})$, any $c \in C_{p,\alpha}(Z \subset \tilde{Z})$ is sent to $C_{p,\varphi(\alpha)}(X \subset Y)$ injectively (because f is surjective), hence $f^*[C_p(Z \subset \tilde{Z})]_\beta = \sum_{\varphi(\alpha)=\beta}[C_p(Z \subset \tilde{Z})]$. Therefore we have

$$\begin{aligned} MC_{p+k}(f^*Z \to Y) &= \sum_{\beta \in B_p(Y)} \left(\sum_{\varphi(\alpha)=\beta} [C_{p,\alpha}(Z \subset \tilde{Z})] \right) t^\beta \\ &= \sum_{\alpha \in B_p(\tilde{Z})} [C_{p,\alpha}(Z \subset \tilde{Z})] t^{\varphi(\alpha)} \end{aligned}$$

which is exactly the image of $\sum_{\alpha \in B_p(\tilde{Z})}[C_{p,\alpha}(Z \subset \tilde{Z})]t^\alpha = MC_p(Z)$ by φ_*.

When for a monic polynomial $g \in R[B_p(\tilde{Z})]$, $MC_p(Z) \cdot g$ is a polynomial, then $\varphi_*(MC_p(Z) \cdot g) = \varphi_*(MC_p(Z)) \cdot \varphi_*(g) = MC_{p+k}(f^*Z \to Y) \cdot \varphi_*(g)$, and as $\varphi_*(g)$ is a monic polynomial in $R[B_p(Y)]$, we have shown that $MC_{p+k}(f^*Z \to Y)$ is rational. □

Corollary 3.16. *Let $X \subset Y$ be a locally closed subscheme of a projective variety Y, and $f : X \to Z$ a flat morphism with relative dimension k to $Z \hookrightarrow C$, an open curve with the projective completion C. When $C \backslash Z = \{P_1, P_2, \dots, P_r\}$ and there exists $\beta \in B_k(Y)$ such that $f^*[P] \in C_{k,\beta}(X \subset Y)$ holds for any $P \in Z$, then $MC_p(f^*Z \to Y)$ is rational for any p. When $r \geq 2$, then $MC_k(f^*Z \to Y) \in K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$ is a polynomial. When moreover Z is a rational curve, then $MC_k(f^*Z \to Y) = (1 - t^\beta)^{r-2} \in K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$.*

Proof. By Proposition 3.15, our configuration is a good configuration in dimension 0. Also it is easy to verify that it is a good configuration in dimension 1, and for all dimensions (except for 0 and 1, the condition is empty). By Kapranov's theorem [10], $MC_0(C)$ is rational, with denominator $(1-t)^2 \in K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$ where t is the class of a point. By Localization Thoerem 3.5, we have $MC_0(Z \subset C)\overline{MC_0(\{P_1, \dots, P_r\})} = MC_0(C)$. As $\overline{MC_0(\{P_1, \dots, P_r\})} = \frac{1}{(1-t)^r}$, we see that $MC_0(Z \subset C)$ is rational, and is a polynomial when $r \geq 2$. When C is $\mathbb{P}^1$, we have $MC_0(Z \subset C) = (1-t)^{r-2}$

and by Proposition 3.15 (2), $MC_k(f^*Z \to Y) = (1-t^\beta)^{r-2}$. For the other p, the equalities $MC_1(Z \subset C) = \frac{1}{1-t^{[C]}}$ and $MC_p(Z \subset C) = 1$, $p \neq 0, 1$ are easy to check. □

4. Torus action

Lemma 4.1. *Let $T = \mathbb{G}_m^n$ be a torus. Assume T acts on a reduced scheme X with the fixed point locus X^T, then we have $[X] = [X^T]$ in $K'(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$.*

Proof. By Thomason's Torus generic slice theorem [16, Prop. 4.10], X can be decomposed into locally closed subschemes $X = \amalg X_i$ such that the stabilizer on X_i is constant, say ${\mathbb{G}_m}^{d_i} \simeq H_i \subset T$, the quotient $X_i/(T/H_i) = \overline{X_i}$ exists and $X_i \simeq \overline{X_i} \times (T/H_i)$. When $d_i < n$, then $[X_i] = 0$ in $K(\mathcal{V}ar/\kappa)_{\mathbb{A}^1}$. In fact, we can write $X_i \simeq Y_i \times \mathbb{G}_m$ where $Y_i = \overline{X_i} \times \mathbb{G}_m^{n-d_i+1}$, then $[X_i] = [Y_i] \times [\mathbb{G}_m] = [Y_i] \times [\mathbb{A}^1] - [Y_i] = 0$. Hence we have $[X] = \sum_{H_i = T}[X_i] = [X^T]$. □

Theorem 4.2. *Assume that X is a reduced scheme on which the torus $T = \mathbb{G}_m^n$ acts. By Thomason's Torus generic slice theorem* [16, Prop. 4.10]*, we can write $X = \amalg X_i$, a decomposition into locally closed subschemes X_i so that the stabilizer on X_i is $H_i \simeq \mathbb{G}_m^{d_i} \subset T$, $X_i/(T/H_i) =: \overline{X_i}$ exists and $X_i \simeq \overline{X_i} \times (T/H_i)$. Let $\pi_i : X_i \to \overline{X_i}$ be the quotient map, then we have*

$$MC_p(X) = \prod_i MC_p(\pi_i^* \overline{X_i} \to X).$$

Proof. We have

$$\begin{aligned} MC_p(X) &= \sum_{\alpha \in B_p(X)} [MC_{p,\alpha}(X)] t^\alpha \\ &= \sum_{\alpha \in B_p(X)} [MC_{p,\alpha}(X)^T] t^\alpha \end{aligned}$$

by Lemma 4.1. A cycle $\sum n_i V_i$ is a fixed point by T if and only if each V_i is fixed by T. An irreducible variety V_i is fixed by T if and only if as a cycle $V_i \in MC_p(\pi_j^* \overline{X_j} \to X)$ for some j.

So for a point $\sum n_i V_i \in C_{p,\alpha}(C)$, it is in $C^T_{p,\alpha}$ if and only if it is in the locus $\prod_k f^*[\overline{X}_{i_k} \subset \tilde{X}_{i_k}]_{\beta_{j_k}}$ with $\sum \beta_{j_k} = \alpha$, for some projective completion $\overline{X}_{i_k} \subset \tilde{X}_{i_k}$ for each k, and once we fix the projective completion $\overline{X}_{i_k} \subset \tilde{X}_{i_k}$, this decomposition is unique, so $[C_{p,\alpha}(X)]$ is the coefficient of t^α in $\prod_i MC_p(\pi_i^* \overline{X}_i \to X)$. □

Definition 4.3. *For an irreducible subvariety $V \subset X$, we define* the degree of V*, denoted as* $\deg V$ *to be the class of V in $B_p(X)$.*

Corollary 4.4. *When X is a toric variety, then we have*

$$MC_p(X) = \prod_V \frac{1}{(1-t^{\deg V})}$$

where V runs over all the p-dimensional orbits.

Proof. As a toric variety X has finitely many orbits by the torus action, so in order to use the formula of Theorem 4.2, we may take X_i to be the set of orbits, where all the quotients $\overline{X_i} = P_i$ are single points. Then as

$$MC_p(\pi_i^* P_i \to X) = \begin{cases} 1 & p \neq \dim X_i \\ \frac{1}{1-t^{\deg X_i}} & p = \dim X_i \end{cases},$$

by Theorem 4.2 we have

$$MC_p(X) = \prod_{\dim X_i = p} \frac{1}{1 - t^{\deg X_i}}$$

□

Example 4.5. Let $P_i \in \mathbb{P}^2$, $(i = 1, 2, 3)$ are three points on $\mathbb{P}^2$, and we assume that they are not colinear. Let X be the blow-up of these three points. We can assume that these points are $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$, and then by the usual torus action, one can regard X as a complete toric variety, whose fan has 1-dimensional cones spanned by $\{(1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1), (0, -1)\}$ respectively. The irreducible divisors corresponding to $(1, 1), (-1, 0), (0, -1)$ are the exceptional divisors, and we denote them as E_1, E_2, E_3, with $P_i \in \mathbb{P}^2$ the image of E_i for each i. Let $\overline{L_i} \subset \mathbb{P}^2$ be the line through P_j, P_k where i, j, k are chosen as $\{i, j, k\} = \{1, 2, 3\}$, and $L_i \subset X$ be the strict transform of $\overline{L_i}$, then L_1, L_2, L_3 are the irreducible T-divisors corresponding to $(-1, -1), (1, 0), (0, 1)$. The effective divisors on X are exactly the linear combination of $[L_1], [L_2], [L_3], [E_1], [E_2], [E_3]$, with non-negative coefficients. The linear equivalence relation is generated by $[L_i] + [E_j] = [L_j] + [E_i]$. We write $t_i := t^{[L_i]}$ and $s_i = t^{[E_i]}$, then by Corollary 4.4, the motivic Chow series of X is

$$MC_1(X) = \frac{1}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - s_1)(1 - s_2)(1 - s_3)}.$$

These 6 variables have the relation $t_i s_j = t_j s_i$.

Let $[L_0] := [L_i] - [E_i]$ (which is the same class for $i = 1, 2, 3$), and let $t_0 := t^{[L_0]}$, then L_0 is not effective, but $\mathbb{Z}[t_1, t_2, t_3, s_1, s_2, s_3] \subset \mathbb{Z}[t_0, s_1, s_2, s_3]$ with $t_i = t_0 s_i$, and the 4 variables $\{t_0, s_1, s_2, s_3\}$ are algebraically independent, and we have another expression of the motivic Chow series as

$$MC_1(X) = \frac{1}{(1 - t_0 s_1)(1 - t_0 s_2)(1 - t_0 s_3)(1 - s_1)(1 - s_2)(1 - s_3)}.$$

Example 4.6. In this example, we compute the motivic Chow series of the blow-up of $\mathbb{P}^2$ along colinear r points.

Let $\overline{L} := \{(0 : y : z)\} \subset \mathbb{P}^2$ be the y-axis, $\{P_1, P_2, \ldots, P_r\} \subset \overline{L}$ be r distinct points on $\overline{L}$. Let $\pi : X \to \mathbb{P}^2$ be the blow-up along $\{P_1, \ldots, P_r\}$, and $L \subset X$ be the strict transform of $\overline{L}$, with $\tilde{P}_i := \pi^{-1}(P_i) \cap L$. Define the action of $\mathbb{G}_m$ on $\mathbb{P}^2$ by $\lambda(x : y : z) := (\lambda x : y : z)$, then this action extends to an action on X uniquely. In this action, the fixed locus $X^{\mathbb{G}_m}$

consists of L and finitely many points, more precisely, $Q_0 := \pi^{-1}(1:0:0)$ and $Q_i \in E_i \backslash L$ where E_i is the exceptional divisor over P_i. Consider the morphism $\lim\limits_{\lambda \to 0} \lambda P : X \to X^{\mathbb{G}_m}$, then $\psi^{-1}(L - \{\tilde{P}_1, \ldots, \tilde{P}_r\}) \cap (X \backslash X^{\mathbb{G}_m}) \simeq (L - \{\tilde{P}_1, \ldots, \tilde{P}_r\}) \times \mathbb{G}_m$. Define $X_0 := \psi^{-1}(L - \{\tilde{P}_1, \ldots, \tilde{P}_r\}) \cap (X \backslash X^{\mathbb{G}_m})$. The other free orbits are $X_i := \psi^{-1} Q_i \cap (X \backslash X^{\mathbb{G}_m})$ and $E_i^\circ := E_i \cap (X \backslash X^{\mathbb{G}_m})$. Write $t^{[L]} =: t_0$ and $t^{[E_i]} =: s_i$. Also let $p_i : X_i \to X_i/\mathbb{G}_m$ and $q_j : E_j^\circ \to E_j^\circ/\mathbb{G}_m$ be the quotient morphisms Then we can write the motivic Chow series of X as

$$
\begin{aligned}
MC_1(X) &= MC_1(X^{\mathbb{G}_m}) \prod_{i=0}^{r} MC_1(p_i^* X_i/\mathbb{G}_m \to X) \prod_{j=0}^{r} MC_0(q_j^* E_i^\circ/\mathbb{G}_m \to X) \\
&= \frac{1}{1-t_0} \cdot (1 - t_0 s_1 s_2 \cdots s_r)^{r-2} \prod_{i=1}^{r} \frac{1}{1 - t_0 s_1 s_2 \cdots \hat{s}_i \cdots s_r} \prod_{j=1}^{r} \frac{1}{1-s_i}
\end{aligned}
$$

where we used Corollary 3.16 for the computation of $MC_1(p_0^* X_0/\mathbb{G}_m \to X)$. In particular when $r = 3$, comparing with Example 4.5, we notice that the motivic Chow series depend on the configuration of the center of the blow-up.

References

[1] Y. André. *Motifs de dimension finite (d'après S. I. Kimura, P. Sullivan...)*, in Séminaire Bourbaki Vol. 2003/2004 Astérisque, vol. 299, Société Mathématique de Frace, 2005, Exp. 929, 115–145

[2] F. Bittner. *The universal Euler characteristic for varieties of characteristic zero.* Compos. Math. 140 1011 – 1032 (2004)

[3] W-L Chow, B. L. van der Wareden. *Zur algebraischen Geometrie IX. Üver zugeordnete Formen und algebraische Systeme von algebraischen Mannigfaltigkeiten*. Math. Ann. 113 (1937) 692 – 704

[4] B. Dwork. *On the rationality of the zeta function of an algebraic variety*, American J. Math. (1960) 631–

[5] J. Elizondo. *The Euler series of restricted Chow varieties*, Compos. Math. 94 (1994) 297–310

[6] E. J. Elizondo, S. Kimura. *Irrationality of Motivic series of Chow varieties*, Math. Z. 263 (2009) 27–32

[7] E. Friedlander. *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. 77 (1991) 55-93

[8] W. Fulton. *Intersection Theory*, Springer (1984)

[9] W. Hoyt. *On the Chow bunches for different projective embeddings of a complete variety*, Amer. J. Math. 88 (1966) 273–278

[10] M. Kapranov: The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups, preprint(arXiv: math.AG/0001005).

[11] S. Kimura. *Chow groups are finite dimensional, in some sense*, Mathematische Annalen **331**, 173-201 (2005)

[12] Larsen, M., Lunts, V. A.: Rationality criteria for motivic zeta functions, Compos. Math. **140**(2004), no. 6, 1537–1560.

[13] D. Mumford, *Abelian Varieties* 2nd ed. Tata Institute of Fundamental Research, Oxford University Press (1974)

[14] M. Nagata, On the normality of the Chow variety of positive 0-cycles of degree m in an algebraic variety, Memoirs of the College of Science, Kyoto, Series A. vol. XXIX (1955) 165–176
[15] T. Scholl, *Classical Motives*, in Proceedings of Symposia in Pure Mathematics Volume 51 (1994), Part 1, 163–187
[16] R. W. Thomason, *Comparison of equivariant algebraic and topological K-theory*, Duke Math 53 (1986) 795–825
[17] A. Weil. *Numbers of solutions of equations in finite fields*, Bulletin of AMS 55 (1949) 497–508
[18] S. Yokura, *Motivic Characteristic classes*, preprint

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